SOLUTE TRANSPORT PROBLEM IN A TWO-DIMENSIONAL FRACTAL POROUS MEDIUM

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Abstract
The problem of anomalous solute transport in a semi-infinite two-dimensional fractal nonhomogeneous porous medium is posed and numerically solved. Initially, the fluid in the medium is considered solute free. Nonhomogeneity of the medium is expressed by linear and quadratic expressions of spatial coordinates for velocity components and diffusion coefficients respectively.

Introduction

That the degradation of air, water and soil has been a major problem, researching solute transport in flow media is demanding great consideration by scientists. This transport of matter can be described by the advection-dispersion equation. It is a partial differential equation in space and time that is of much significance in such diverse disciplines as chemical and petroleum engineering or bio and soil physics [1]. For example, the advection-dispersion equation can be used to determine the pollutant concentration downstream from intended mining operations in order to predict and plan how to reduce their environmental footprint.

Lindstrom and Boersma [2] have researched analytical solutions for one-dimensional solute transport through media idealized as homogeneous. However, the actual solute permeation through air, soil or groundwater tends to be position dependent. To account for this heterogeneity, spatially-dependent dispersion and velocity should be considered. This has been solved analytically for special cases in one dimension [3–4]. Numerical solutions are required for cases that are more general and for problems in two or three dimensions [5–6]. Dehghan [7] employed weighed explicit finite difference method (EFDM) for one-dimensional advection-dispersion equation with increased accuracy of the obtained numerical results if compared to that of standard finite difference methods. Karahan [8] employed implicit finite difference method (IFDM) for one-dimensional advection-dispersion equation using spreadsheets. Walter et al. [9] used Crank-Nicholson central difference scheme in one dimension to model soil solute release into runoff with infiltration. Thanks to being unconditionally stable IFDMs can give great opportunity to use larger step lengths though extremely large matrices must be manipulated. For this reason, IFDMs is more efficient than EFDMs because by using IFDMs calculations can be manipulated at any time and coordinate step lengths. EFDM is also simpler in addition to being computationally more efficient. S. Savovic and A. Djordjevich have solved the problem of one-dimensional advection-dispersion equation with variable coefficients by using EFDM [10] and they expanded their work by solving the two-dimensional ADE in semi-finite media [11]. In this paragraph two-dimensional ADE with variable coefficients in semi-finite media is solved using IFDM and the problem is considered in fractal porous media in next paragraph. Dispersion unsteadiness is another variation that is allowed in order to accommodate the finding by Freeze and Cherry [12] that the dispersion is proportional to the n th power of velocity, with the exponent n ranging from 1 to 2 [11].

Mathematical model of the problem

Let solute particles of a pollutant be entering a body of air, soil or water (including groundwater) at uniform rate at some location, continuously for a fixed amount of time. In other words, there is a stationary point-source emitting a uniform pulse of pollutants (Fig. 2.1). This could be a smokestack, volcano, sewage outlet, or infiltration from a garbage dump, septic tank or tailings pond that is uniformly active for a fixed period of time. From such point-source as the origin of mutually perpendicular horizontal x and y axes $\left(0 \le x < \infty;\ 0 \le y < \infty\right)$ defining a horizontal plane, solute particles are transported by diffusion and convection mainly downstream in the longitudinal direction chosen for the x-axis (with the y-axis along the transverse direction) [11].

Let the velocity components of the flow field in x and y directions at position (x,y) in the horizontal plane be $u(x,t)$ and $v(y,t)$, respectively. Both satisfy the Darcy law if the medium is porous; or laminar flow conditions otherwise. Further, let $D_x(x,t)$ and $D_y(y,t)$ be longitudinal and transverse components of the solute dispersivity parameter at the same position, respectively [67]. The linear advection-dispersion partial differential equation in two-dimensional horizontal plane medium may be written in the following general form [11]:

Fractional advection dispersion equation can be written in the following form

$$\frac{\partial C(x,y,t)}{\partial t}=\frac{\partial^{\beta_1}}{\partial x}\left(D_x(x,t)\frac{\partial C(x,y,t)}{\partial x}-u(x,t)C(x,y,t)\right)+ +\frac{\partial^{\beta_2}}{\partial y}\left(D_y(y,t)\frac{\partial C(x,y,t)}{\partial y}-v(y,t)C(x,y,t)\right) \quad (1)$$

where C is solute concentration, D_x and D_y are diffusion coefficients, u and v velocity components, $0\le\alpha\le1$, $0\le\beta_1\le1$ and $0\le\beta_2\le1$ are orders of fractional derivatives with respect to t, x and y respectively.

Here we used exponential functions to express diffusion and velocity components [11].

$$\begin{aligned} D_x &= D_{x0}\, f_2(mt)\cdot(1+ax)^2 \\ D_y &= D_{y0}\, f_2(mt)\cdot(1+by)^2 \\ u_x &= u_0\, f_1(mt)\cdot(1+ax) \\ v_y &= v_0\, f_1(mt)\cdot(1+by) \end{aligned} \quad (2)$$

where D_{x0} and D_{y0} are initial diffusion coefficients at the $(0,0)$ point. u_0 and v_0 are initial velocity components at the $(0,0)$ point. $f_1(mt)=\exp(-mt)$ and $f_2(mt)=\exp(mt)$ are exponential given functions, a and b are nonhomogeneity coefficients. m is unsteadiness coefficient of flow.

The numerical algorithm

We can rewrite the equation (1) in the following form

$$\frac{\partial C(x,y,t)}{\partial t}=\frac{\partial^{\beta_1}D_x(x,t)}{\partial x^{\beta_1}}\cdot\frac{\partial C(x,y,t)}{\partial x}+D_x(x,t)\cdot\frac{\partial^{1+\beta_1}C(x,y,t)}{\partial x^{1+\beta_1}}- -\frac{\partial^{\beta_1}u(x,t)}{\partial x^{\beta_1}}\cdot C(x,y,t)-u(x,t)\cdot\frac{\partial^{\beta_1}C(x,y,t)}{\partial x^{\beta_1}}+\frac{\partial^{\beta_2}D_y(y,t)}{\partial y^{\beta_2}}\cdot\frac{\partial C(x,y,t)}{\partial y}+ +D_y(y,t)\cdot\frac{\partial^{1+\beta_2}C(x,y,t)}{\partial y^{1+\beta_2}}-\frac{\partial^{\beta_2}v(y,t)}{\partial y^{\beta_2}}-v(y,t)\cdot\frac{\partial^{\beta_2}C(x,y,t)}{\partial y} \quad (3)$$

In order to solve the fractional advection dispersion equation we use Caputo fractional derivative formula [3]. We used following set function to approximate equation (3) [13].

$$\Omega=\{(x,y,t):0\le x\le L_1\ ,0\le y\le L_2,\ \ 0\le t\le\tau\}$$

$$\Omega_{nj}=\left\{\left(x_i,y_j,t_k\right):x_i=i\cdot h_1,\ i=\overline{1,\mathrm{N}};\ y_j=j\cdot h_2,\ j=\overline{1,\mathrm{M}};\right.$$

$$\left.t_k=k\cdot\tau,\ k=\overline{1,\mathrm{K}},h_1=\frac{X}{N},h_2=\frac{Y}{M},\tau=\frac{T}{K}\right\}$$

We approximated (3) equation using explicit finite difference scheme [13].

$$\frac{C_{i,j}^{k+1}-C_{i,j}^{k}}{\tau}=\frac{(D_x)_i^k-\beta_1(D_x)_{i-1}^k}{\Gamma(2-\beta_1)h_1^{\beta_1}}\cdot\frac{C_{i,j}^k-C_{i-1,j}^k}{h_1}+(D_x)_i^k\cdot\frac{1}{\Gamma(3-\gamma_1)\cdot h_1^{\gamma_1}}\cdot$$
$$\sum_{l=0}^{i-1}\left[\left(C_{i+1-l,j}^k-2\cdot C_{i-l,j}^k+C_{i-1-l,j}^k\right)\times\left((l+1)^{2-\gamma_1}-l^{2-\gamma_1}\right)\right]-C_{i,j}^k\cdot\frac{u_i^k-\beta_1 u_{i-1}^k}{\Gamma(2-\beta_1)\cdot h_1^{\beta_1}}-$$
$$-u_i^k\cdot\frac{C_{i,j}^k-\beta_1\cdot C_{i-1,j}^k}{\Gamma(2-\beta_1)\cdot h_1^{\beta_1}}+\frac{(D_y)_j^k-\beta_2\cdot(D_y)_{j-1}^k}{\Gamma(2-\beta_2)\cdot h_2^{\beta_2}}\cdot\frac{C_{i,j}^k-C_{i,j-1}^k}{h_2}+\frac{(D_y)_j^k}{\Gamma(3-\gamma_2)\cdot h_2^{\gamma_2}}\cdot \tag{4}$$
$$\sum_{l=0}^{j-1}\left[\left(C_{i,j+1-l}^k-2\cdot C_{i,j-l}^k+C_{i,j-1-l}^k\right)\times\left((l+1)^{2-\gamma_1}-l^{2-\gamma_1}\right)\right]-C_{i,j}^k\cdot\frac{v_j^k-\beta_2 v_{j-1}^k}{\Gamma(2-\beta_2)\cdot h_2^{\beta_2}}-$$
$$-v_j^k\frac{C_{i,j}^k-\beta_2\cdot C_{i,j-1}^k}{\Gamma(2-\beta_2)\cdot h_2^{\beta_2}}$$

Equation (4) can be written in the following form

$$C_{i,j}^{k+1}=\frac{\tau\cdot\left((D_x)_i^k-\beta_1(D_x)_{i-1}^k\right)\cdot\left(C_{i,j}^k-C_{i-1,j}^k\right)}{h_1^{1+\beta_1}\cdot\Gamma(2-\beta_1)}+\frac{\tau\cdot(D_x)_i^k}{\Gamma(3-\gamma_1)\cdot h_1^{\gamma_1}}\cdot$$
$$\cdot\sum_{l=0}^{i-1}\left[\left(C_{i+1-l,j}^k-2\cdot C_{i-l,j}^k+C_{i-1-l,j}^k\right)\times\left((l+1)^{2-\gamma_1}-l^{2-\gamma_1}\right)\right]-\frac{\tau\cdot C_{i,j}^k\left(u_i^k-\beta_1 u_{i-1}^k\right)}{\Gamma(2-\beta_1)\cdot h_1^{\beta_1}}-$$
$$-\frac{\tau\cdot u_i^k\left(C_{i,j}^k-\beta_1\cdot C_{i-1,j}^k\right)}{\Gamma(2-\beta_1)\cdot h_1^{\beta_1}}+\frac{\tau\cdot\left((D_y)_j^k-\beta_2\cdot(D_y)_{j-1}^k\right)\cdot\left(C_{i,j}^k-C_{i,j-1}^k\right)}{\Gamma(2-\beta_2)\cdot h_2^{1+\beta_2}}+ \tag{5}$$
$$+\frac{\tau\cdot(D_y)_j^k}{\Gamma(3-\gamma_2)\cdot h_2^{\gamma_2}}\cdot\sum_{l=0}^{j-1}\left[\left(C_{i,j+1-l}^k-2\cdot C_{i,j-l}^k+C_{i,j-1-l}^k\right)\times\left((l+1)^{2-\gamma_1}-l^{2-\gamma_1}\right)\right]-$$
$$-\tau\cdot C_{i,j}^k\cdot\frac{v_j^k-\beta_2 v_{j-1}^k}{\Gamma(2-\beta_2)\cdot h_2^{\beta_2}}-\tau\cdot v_j^k\frac{C_{i,j}^k-\beta_2\cdot C_{i,j-1}^k}{\Gamma(2-\beta_2)\cdot h_2^{\beta_2}}+C_{i,j}^k$$

Equation (5) is a recurrent formula and the next values of concentration is calculated step by step. So as to solve the equation we need to include initial and boundary conditions.

$$C_{i,j}^0=0,\qquad x\geq 0;\ \ y\geq 0;\ \ t=0 \tag{6}$$

Equation (6) is the initial condition. Boundary conditions are calculated in the x and y directions in the following form.

$$C_{0,j}^{k+1}=\frac{\tau\cdot\left(\left(D_x\right)_1^k-\beta_1\left(D_x\right)_0^k\right)\cdot\left(C_{1,j}^k-C_{0,j}^k\right)}{h_1^{1+\beta_1}\cdot\Gamma\left(2-\beta_1\right)}-\frac{\tau\cdot C_{0,j}^k\left(u_1^k-\beta_1 u_0^k\right)}{\Gamma\left(2-\beta_1\right)\cdot h_1^{\beta_1}}+$$
$$+\frac{\tau\cdot\left(\left(D_y\right)_j^k-\beta_2\cdot\left(D_y\right)_{j-1}^k\right)\cdot\left(C_{0,j}^k-C_{0,j-1}^k\right)}{\Gamma\left(2-\beta_2\right)\cdot h_2^{1+\beta_2}}+\frac{\tau\cdot\left(D_y\right)_j^k}{\Gamma\left(3-\gamma_2\right)\cdot h_2^{\gamma_2}}\cdot \qquad (7)$$
$$\sum_{l=0}^{j-1}\left[\left(C_{0,j+1-l}^k-2\cdot C_{0,j-l}^k+C_{0,j-1-l}^k\right)\times\left(\left(l+1\right)^{2-\gamma_1}-l^{2-\gamma_1}\right)\right]-$$
$$-\tau\cdot C_{0,j}^k\cdot\frac{v_j^k-\beta_2 v_{j-1}^k}{\Gamma\left(2-\beta_2\right)\cdot h_2^{\beta_2}}-\tau\cdot v_j^k\frac{C_{0,j}^k-\beta_2\cdot C_{0,j-1}^k}{\Gamma\left(2-\beta_2\right)\cdot h_2^{\beta_2}}+C_{0,j}^k$$

$$C_{i,0}^{k+1}=\frac{\tau\cdot\left(\left(D_x\right)_i^k-\beta_1\left(D_x\right)_{i-1}^k\right)\cdot\left(C_{i,0}^k-C_{i-1,0}^k\right)}{h_1^{1+\beta_1}\cdot\Gamma\left(2-\beta_1\right)}+\frac{\tau\cdot\left(D_x\right)_i^k}{\Gamma\left(3-\gamma_1\right)\cdot h_1^{\gamma_1}}\cdot$$
$$\cdot\sum_{l=0}^{i-1}\left[\left(C_{i+1-l,0}^k-2\cdot C_{i-l,0}^k+C_{i-1-l,0}^k\right)\times\left(\left(l+1\right)^{2-\gamma_1}-l^{2-\gamma_1}\right)\right]-$$
$$-\frac{\tau\cdot C_{i,0}^k\left(u_i^k-\beta_1 u_{i-1}^k\right)}{\Gamma\left(2-\beta_1\right)\cdot h_1^{\beta_1}}-\frac{\tau\cdot u_i^k\left(C_{i,0}^k-\beta_1\cdot C_{i-1,0}^k\right)}{\Gamma\left(2-\beta_1\right)\cdot h_1^{\beta_1}}+ \qquad (8)$$
$$+\frac{\tau\cdot\left(\left(D_y\right)_1^k-\beta_2\cdot\left(D_y\right)_0^k\right)\cdot\left(C_{i,1}^k-C_{i,0}^k\right)}{\Gamma\left(2-\beta_2\right)\cdot h_2^{1+\beta_2}}-\tau\cdot C_{i,1}^k\cdot\frac{v_1^k-\beta_2 v_0^k}{\Gamma\left(2-\beta_2\right)\cdot h_2^{\beta_2}}+C_{i,j}^k$$

(7) and (8) are boundary conditions where $i=\overline{1,N-1}$, $j=\overline{1,R-1}$, $N=\dfrac{x_\infty}{h_1}$, $R=\dfrac{y_\infty}{h_2}$ are the grid dimension in the x and y directions, respectively, x_∞ is the distance in direction x at which $\dfrac{\partial C}{\partial x}=0$ and y_∞ is the distance in direction y at which $\dfrac{\partial C}{\partial y}=0$, and $C_{i,R}^{k+1}=\beta_2\cdot C_{i,R-1}^{k+1}$; $C_{N,j}^{k+1}=\beta_1\cdot C_{N-1,j}^{k+1}$
.

Numerical analysis

After solving the initial-boundary problem we obtained following results.

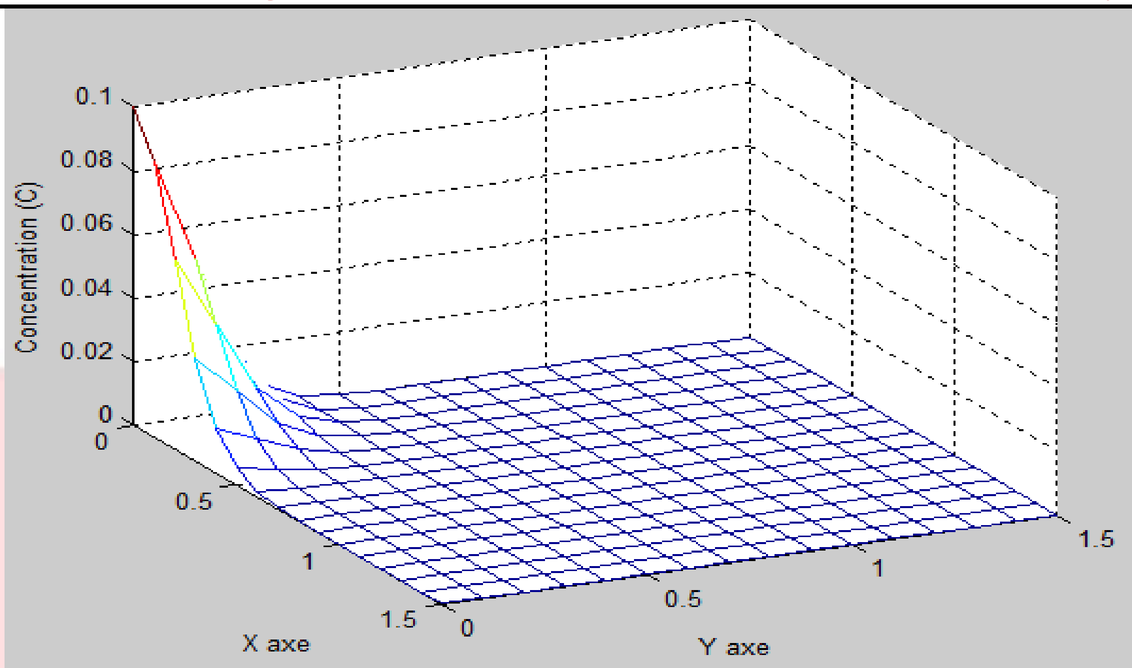

Fig. 1. Concentration field where time limit $T = 1000\ s$, $X = 1{,}5\ m$, $Y = 1{,}5\ m$. The orders of derivative with respect to x and y directions $\beta_1 = 1, \beta_2 = 1$.

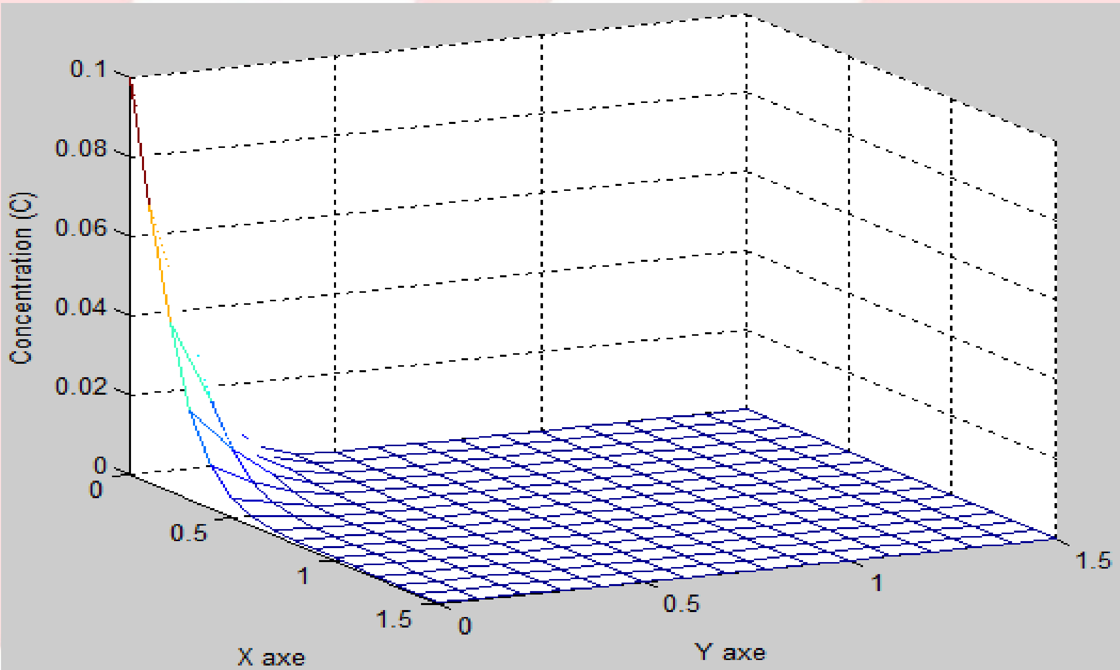

Fig. 2. Concentration field where time limit $T = 1000\ s$, $X = 1{,}5\ m$, $Y = 1{,}5\ m$. The orders of derivative with respect to x and y directions $\beta_1 = 0.9$; $\beta_2 = 0.9$.

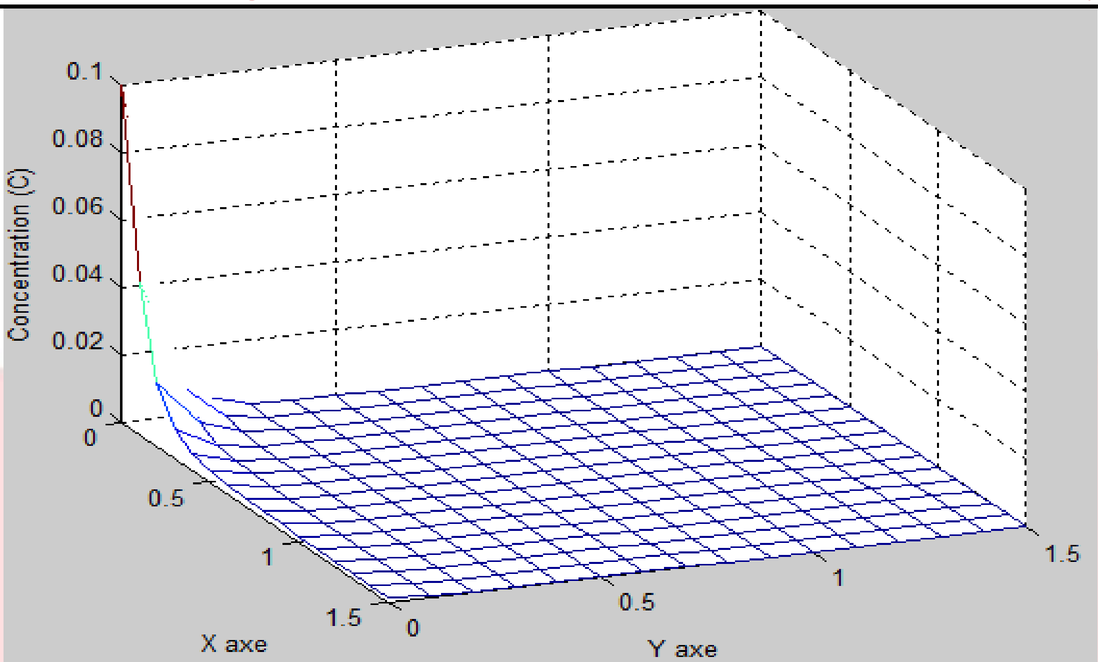

Fig. 3. Concentration field where time limit $T = 1000\ s$, $X = 1{,}5\ m$, $Y = 1{,}5\ m$. The orders of derivative with respect to x and y directions $\beta_1 = 0.7$; $\beta_2 = 0.7$.

The results are compared in figures of 1-3 by using different fractional orders of derivatives with respect to coordinate components. By considering the figures it can be seen that decreasing the order of derivatives leads to wider spreading of concentration profiles.

Conclusion

The problem of filtering and transporting suspension in a two-dimensional porous medium with a fractal structure is considered. Solute transport in such media is described by an equation with fractional derivatives both in time and in coordinate. On the basis of the numerical solution of the equation with the corresponding initial and boundary conditions, it was shown that a decrease in the fractional derivative with respect to time, weakens the transport process. A decrease in the order of the fractional derivative with respect to coordinates, on the contrary, accelerates the diffusion process, i.e. the “fast diffusion” effect appears.

References.

1. Rao C.S. Environmental Pollution Control Engineering, 3rd reprint., Wiley Eastern Ltd, New Delhi, 1995.
2. Lindstrom F.T., Boersma L. Analytical solutions for convective-dispersive transport in confined aquifers with different initial and boundary conditions, Water Resour. Res. 15 1989, 241–256.
3. Ebach E.H., White R. Mixing of fluids flowing through beds of packed solids, J. Am. Inst. Chem. Eng. 4 (1958) 161–164
4. Perez Guerrero J.S., Pimentel L.C.G., Skaggs T.H. Analytical solution of the advection–dispersion transport equation in layered media, Int. J. Heat Mass Transfer 56 (2013) 274–282.
5. Ciftci E., Avci C.B., Borekci O.S., Sahin A.U. Assessment of advective–dispersive contaminant transport in heterogeneous aquifers using a meshless method, Environ. Earth Sci. 67 (2012) 2399–2409.
6. Zhao C., Valliappan S., Numerical modelling of transient contaminant migration problems in infinite porous fractured media using finite/infinite element technique: theory, Int. J. Numer. Anal. Methods Geomech. 18 (1994) 523–541.
7. Dehghan M. Weighted finite difference techniques for the one-dimensional advection–diffusion equation, Appl. Math. Comput. 147 (2004) 307–319.
8. Karahan H. Implicit finite difference techniques for the advection–diffusion equation using spreadsheets, Adv. Eng. Software 37 (2006) 601–608.
9. Gnedenko, B. V. and Kolmogorov, A. N. Limit Distributions for Sums of Independent Random Variables (Addison-Wesley, Cambridge, MA) 1954.

10. Savovic S., Djordjevich A. Finite difference solution of the one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media, Int. J. Heat Mass Transfer 55 (2012) 4291-4294.
11. Savovic S., Djordjevich A. Numerical solution for temporally and spatially dependent solute dispersion of pulse type input concentration in semi-infinite media, Int. J. Heat Mass Transfer 60 (2013) 291–295.
12. Freeze R.A., Cheery J.A. Groundwater, Prentice-Hall, New Jersey, 1979.
13. A.Samarskii. The Theory of difference schemes. Taylor & Francis Inc., New York, United States, 2001-788 Pp.